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A General Descent Framework for the Monotone Variational Inequality Problem

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Abstract

We present a framework for descent algorithms that solve the monotone variational inequality problem VIP_v which consists in finding a solution $v \epsilon \Omega_v$ satisfying $s(v)^T (u - v) \geq 0$, for all $u \epsilon \Omega_v$. This unified framework includes, as special cases, some well-known iterative methods and equivalent optimization formulations. A descent method is developed for an equivalent general optimization formulation and a proof of its convergence is given. Based on this unified algorithmic framework, we show that a variant of the descent method, where each subproblem is only solved approximately, is globally convergent under certain conditions.

Keywords: Variational Inequality Problem, Descent Methods, Optimization.

Résumé

Nous présentons un cadre théorique pour des algorithmes de descente pour résoudre le problème d'inégalité variationnelle monotone VIP_v, soit de trouver une solution $v \epsilon \Omega_v$ qui satisfait $s(v)^T (u - v) \geq 0$, pour tout $u \epsilon \Omega_v$. Ce cadre unifié inclut, comme cas particuliers, quelques méthodes itératives connues ainsi que des formulations équivalentes comme problème d'optimisation. Une méthode de descente est développée pour une formulation équivalente générale comme problème d'optimisation et une preuve de convergence est donnée. Basé sur ce cadre théorique unifié, nous démontrons qu'une variante de la méthode de descente, où chaque sous-problème n'est résolu qu'approximativement, converge globalement sous certaines conditions.

Mots-clés: Problème d'inégalité variationnelle, Méthodes de descente, Optimisation.

1 Introduction

The variational inequality problem VIP is defined as follows:

find $v \in \Omega_v$ such that

$$(VIP_v) \qquad s(v)^T(u-v) \geq 0, \quad \text{for all} \quad u \in \Omega_v \tag{1}$$

or alternatively let

$$v = \delta h, \qquad S(h) = \delta^T s(v) \tag{2}$$

find $h \in \Omega_h$ such that

$$(VIP_h) \qquad s(\delta h)^T \delta(u-h) \geq 0, \quad \text{for all} \quad h \in \Omega_h \tag{3}$$

where $\Omega_v \subset R^m$ and $\Omega_h \subset R^n$ are both convex and compact sets;

$v \in R^m$ and $h \in R^n$ are both vectors ;
$s(v) : \Omega_v \subset R^m \rightarrow R^m$ is a mapping;
$S(h) : \Omega_h \subset R^n \rightarrow R^n$ is a mapping;
δ is an $m \times n$ linear transformation matrix;
T denotes transposition.

It is well known that when $\nabla s(v)$ is symmetric and positive definite on Ω_v, VIP_v corresponds to the optimality condition of a convex program; if $\nabla s(v)$ does not satisfy the condition, then there is no such convex program.

Both VIP_v and VIP_h have been widely used to model network equilibrium problems arising in transportation system analysis (Smith, 1979, 1983, Dafermos, 1980, Bertsekas and Gafni,1982, Florian and Spiess, 1984, Florian, 1986, Marcotte and Guélat, 1988) and regional science (Florian and Los, 1982, Nagurney, 1987, Dafermos and Nagurney, 1987). The study of variational inequalities draws on nonlinear optimization methodology, solution of nonlinear equations and fixed point theory.

In order to solve VIP_v or VIP_h, two main approaches have emerged in the recent literature: the iterative approach, where the solution approach is based on solving a sequence of simpler problems (optimization problems, linear or nonlinear complementarity problems, such as Pang and Chan, 1982), and the solution of an equivalent optimization problem where the variational inequality is solved as a mathematical program (both differentiable and nondifferentiable, such as Nguyen and Dupuis, 1984, Marcotte, 1985,

Dussault and Marcotte, 1985). The first approach draws on the theory of the iterative solution of nonlinear systems of equations (Ortega and Rheinboldt, 1970) and includes algorithms based on SOR, Gauss-Seidel, Newton, linearized Jacobi and projection methods. Among others, Bertsekas and Gafni (1982) present a projection method for VIP_h and Dafermos (1982, 1983) presents a general iterative scheme including the projection method for VIP_v as a special case. Nguyen and Dupuis (1984) propose a cutting plane approach for VIP_v. Marcotte (1985) proposes an algorithm for the solution of an equivalent nondifferentiable optimization formulation of VIP_v for monotone $s(v)$, while Marcotte and Dussault(1990) solve an equivalent optimization formulation for VIP_v by solving a sequence of linear programs. Fukushima (1986) proposes a modified projection method for VIP_v, where each iteration requires a projection onto a supporting half space, rather than just the solution space itself. Harker (1988) developed an acceleration method for the Jacobi and projection algorithms for solving monotone VIP_v which is reminiscent of the PARTAN variant for primal optimization algorithms. See Harker and Pang (1990) for a comprehensive survey of the subject.

A significant contribution to the reformulation of VIP_v as an equivalent differentiable optimization problem is that of Fukushima (1989) who develops a continuously differentiable but nonconvex equivalent optimization formulation for VIP_v when $\nabla s(v)$ is asymmetric, by using the projection operator. We shall refer to this formulation as the projection based optimization model (PBOM). Motivated by Fukushima's work, Wu, Florian and Marcotte (1990) consider a more general reformulation for VIP_h, which we refer to as the linearized Jacobi based optimization model (LJBOM). Pang (1990) studies the Newton method for solvinga system of B-differentiable equations and applies the resulting method to the nonlinear complementarity, variational inequality and nonlinear programming prolems.

In this paper, we consider only VIP_v and present a general equivalent optimization formulation for VIP_v. Based on the formulation, a general descent framework is developed for the general equivalent optimization formulation and a proof of its convergence is given. Then we show that the symmetrized Newton method, Quasi-Newton methods, the linearized Jacobi method, the projection method as well as others are various realizations of this general algorithmic framework. In addition, we show that a variant of the descent method, where each subproblem is only solved approximately, is globally convergent under certain conditions. The theoretical development parallels that of Migdalas (1990) for the solution of convex optimization

problem by a regularized Frank-Wolfe (1956) algorithm. Although we only consider VIP_v in this paper, it is important to note that the results may be extended to VIP_h, as analysed in Wu, Florian and Marcotte (1990).

This paper is organized as follows. In section 2, we give some definitions and assumptions. In section 3, a general equivalent optimization formulation is presented for VIP_v with a full discussion of its properties. Then, a general descent method is developed for this formulation. Section 4 is devoted to an analysis of the relationship with some other existing algorithms, while, in section 5, we consider a possible way of applying an approximation strategy to the general framework in order to derive an approximate algorithm. Section 6 provides some conclusions.

2 Definitions and assumptions

In this section, we give some definitions and assumptions that are used in our analysis.

The mapping $s(v) : R^m \rightarrow R^m$ is said to be monotone on Ω_v if

$$(s(v') - s(v''))^T(v' - v'') \geq 0 \quad \text{for all } v', v'' \in \Omega_v, \tag{4}$$

strictly monotone on Ω_v if strict inequality holds in (4) for $v' \neq v''$ and strongly monotone on Ω_v if there is a positive λ such that

$$(s(v') - s(v''))^T(v' - v'') \geq \lambda||v' - v''||_2^2 \quad \text{for all } v', v'' \in \Omega_v \tag{5}$$

It is well known (Ortega and Rheinboldt, 1970) that, if $s(v)$ is continuously differentiable and the Jacobian matrix $\nabla s(v)$ is positive definite on Ω_v,

$$(v'-v'')^T \nabla s(v)(v'-v'') \leq \lambda||v'-v''||_2^2 \text{ for all } , v', v'' \in \Omega_v, \ v' \neq v'' \tag{6}$$

then s is strongly monotone on Ω_v with λ the minimum eigenvalue of the symmetric part of $\nabla s(v)$ over Ω_v, which we denote λ_{min}. Note that $\nabla s(v)$ need not be symmetric in (6).

If the mapping $s(v) : R^m \rightarrow R^m$ is Lipschitz continuous on Ω_v, then there is a constant L_s such that

$$||s(v') - s(v'')||_2 \leq L_s||v' - v''||_2 \quad \text{for all } v', v'' \in \Omega_v. \tag{7}$$

3 A general equivalent optimization formulation

In this section, we give a general equivalent optimization formulation for VIP_v and discuss its properties.

Consider the following optimization problem:

$$\min_v f(v) = -s(v)^T(H(v) - v) - \frac{1}{\alpha}\phi(H(v), v) \tag{8}$$

$$\text{s.t. } v \in \Omega_v$$

$$H(v) = \arg\min_u s(v)^T(u - v) + \frac{1}{\alpha}\phi(u, v)$$

$$\text{s.t. } u \in \Omega_v$$

where $\phi(u, v) : \Omega_v \times \Omega_v \subset R^m \to R^1$ has the following properties:

$-$it is continuously differentiable on $\Omega_v \times \Omega_v$; (9)

$-$it is nonnegative on $\Omega_v \times \Omega_v$; (10)

$-$it is strictly convex on Ω_v; (11)

$-\phi(u, v) = 0 \Leftrightarrow u = v$; (12)

$-$its Hessian matrix with respect to

u is symmetric, positive definite on Ω_v; (13)

$-\ \alpha$ is a positive parameter. (14)

Note that if $\alpha = \infty$, then $H(v)$ may be a point-to-set mapping.
The following several possible forms for $\phi(u, v)$ satisfy (9)-(14):

$$\phi(u, v) = \frac{1}{2}||u - v||_2^2 \tag{15}$$

$$\phi(u, v) = \frac{1}{2}(u - v)^T B(u - v), \tag{16}$$

where B is a symmetric and positive definite matrix,

$$\phi(u, v) = \frac{1}{2}(u - v)^T B(v)(u - v) \tag{17}$$

where $B(v)$ is a symmetric and positive definite matrix.

Note that $\phi(u,v)$ is a proximity measure between u and v.

Now, consider the following subproblem of (8) for a fixed v:

$$\min_u \psi(u,v) = s(v)^T(u-v) + \frac{1}{\alpha}\phi(u,v) \qquad (18)$$
$$\text{s.t.} \quad u \in \Omega_v.$$

It is clear that the program (18) is strictly convex and hence is well defined for any α such that $0 < \alpha < \infty$. Furthermore, $\psi(u,v)$ is also a proximity measure between u and v. If $\alpha = \infty$, then the solution to the linear program (18) may not be unique.

Proposition 3.1 *Assume $\alpha > 0$ and let $H(v)$ be any solution of (18). Then (18) is equivalent to*

$$[s(v) + \frac{1}{\alpha}\nabla_u \phi(H(v),v)]^T(u - H(v)) \geq 0 \quad \text{for all} \quad u \in \Omega_v. \qquad (19)$$

Proof. Case 1: $0 < \alpha < \infty$. The variational inequality defined in (19) is in fact a necessary and sufficient optimality condition of the convex program (18), since the Hessian matrix of $\phi(u,v)$with respect of u is symmetric positive definite.

Case 2: $\alpha = \infty$. $s(v)^T(u - H(v))$ is known as the gap function (Marcotte, 1985) which is always greater than or equal to zero, i.e., $s(v)^T u \geq s(v)^T H(v)$. Thus $H(v)$ is a solution (18) and vice versa. □

Proposition 3.2 *$H(v)$ has a fixed point $v \in \Omega_v$, i.e., $v \in H(v)$.*

Proof. Since $H(v)$ is an upper semicontinuous point-to-set mapping (continuous, if $0 < \alpha < \infty$), the result follows from Kakutani's fixed point theorem (1964). □

Proposition 3.3 *A vector v solves the variational inequality problem VIP_v if and only if it is a fixed point of the mapping H, i.e., $v \in H(v)$.*

Proof. See Dafermos (Lemma 1, 1982). □

Proposition 3.4

$$f(v) \text{ is Lipschitz continuous on } \Omega_v \quad . \qquad (20)$$

$$\partial f(v) = \mathrm{Co}_{u \in H(v)}[s(v) - \nabla s(v)(u - v) - \frac{1}{\alpha} \nabla_v \phi(u, v)]. \tag{21}$$

where $H(v)$ is the solution set of the subproblem (18).

$$f(v) \text{ is differentiable at } v \text{ if } 0 < \alpha < \infty. \tag{22}$$

Proof. See Clarke (1975) for (20) and (21). For (22), note that $0 < \alpha < \infty$ ensures uniqueness of $H(v)$ by Danskin's theorem (1966). We then have that $f(v)$ is differentiable. □.

Theorem 3.1 *$f(v) \geq 0$ for all $v \in \Omega_v$ and $f(v) = 0$, i.e., v solves (8), if and only if v solves VIP_v.*

Proof. For the first part, since $f(v)$ is the negative of the optimal value of subproblem (18) which is less or equal to zero (take $u = v$), it is clear that $f(v) \geq 0$ for every $v \in \Omega_v$.

For the second part, suppose that v solves VIP_v. Then by Proposition 3.3, we have $v \in H(v)$, which implies $f(v) = 0$.

Conversely suppose that $f(v) = 0$. It then follows from $f(v)$ that

$$s(v)^T(H(v) - v) = -\frac{1}{\alpha}\phi(H(v), v). \tag{23}$$

Since $s(v)^T(H(v) - v) + \frac{1}{\alpha}(H(v), v) \leq 0$, we have

$$s(v)^T(H(v) - v) \leq -\frac{1}{\alpha}\phi(H(v), v). \tag{24}$$

Combining (23) and (24), we have $v \in H(v)$. By Proposition 3.3, v is then a solution of VIP_v. □

Proposition 3.5 *Assume that $s(v)$ is strongly monotone on Ω_v and $\nabla_u \phi(u, v) = -\nabla_v \phi(u, v)$. Suppose that v is not a solution of VIP_v. Then there exists $u \in H(v)$ such that $d = u - v$ is a descent direction for f at v, i.e.,*

$$f'(v; d) < 0,$$

more precisely

$$f'(v; d) \leq -f(v), \text{ if } \alpha = \infty, \tag{25}$$

$$f'(v;d) \leq -\lambda_{min}||d||_2^2, \quad \text{if } 0 < \alpha < \infty. \tag{26}$$

Proof.
Case 1: $\alpha = \infty$. We have $f'(v;d) \leq -f(v)$ (see Marcotte, 1985).
Case 2: $0 < \alpha < \infty$. We have

$$\begin{aligned} f'(v;d) &= -(u-v)^T \bigtriangledown s(v)(u-v) + [s(v) - \frac{1}{\alpha} \bigtriangledown_v \phi(u,v)]^T(u-v) \\ &\leq -\lambda_{min}||d||_2^2 + [s(v) + \frac{1}{\alpha} \bigtriangledown_u \phi(u,v)]^T(u-v) \\ &\leq -\lambda_{min}||d||_2^2 - [s(v) + \frac{1}{\alpha} \bigtriangledown_u \phi(u,v)]^T(v-u) \\ &\leq -\lambda_{min}||d||_2^2 \qquad \text{by (19).} \quad \square \end{aligned}$$

If the form (15) or (16) of $\phi(u,v)$ is used, then the second condition of Proposition 3.5 is satisfied.

Proposition 3.6 *Assume that $s(v)$ is strongly monotone on Ω_v and the matrix $\bigtriangledown B(v)(H(v)-v)$ is positive definite on Ω_v with minimum eigenvalue β_{min}, if the form (17) of $\phi(u,v)$ is used. Let $d = u - v$ where $u \in H(v)$. If $d \neq 0$, then there exists $u \in H(v)$ such that d is a descent direction for f at v, i.e.,*

$$f'(v;d) < 0,$$

more precisely:

$$f'(v;d) \leq -f(v), \quad \text{if } \alpha = \infty, \tag{27}$$

$$f'(v;d) \leq -(\lambda_{min} + \frac{\beta_{min}}{2\alpha})||d||_2^2, \quad \text{if } 0 < \alpha < \infty. \tag{28}$$

Proof.
Case 1: $\alpha = \infty$. We have $f'(v;d) \leq -f(v)$ (see Marcotte, 1985).
Case 2: $0 < \alpha < \infty$. We have

$$\begin{aligned} f'(v;d) = (u-v)^T[s(v) - (\bigtriangledown s(v) - \frac{B(v)}{\alpha})(u-v) \\ -\frac{1}{2\alpha}(u-v)^T \bigtriangledown B(v)(u-v)] \end{aligned}$$

$$= [s(v) + \frac{B(v)}{\alpha}(u-v)]^T(u-v) - (u-v)^T \bigtriangledown s(v)(u-v)$$

$$-\frac{1}{2\alpha}(u-v)^T \bigtriangledown B(v)(u-v)(u-v)$$

$$\leq -(u-v)^T \bigtriangledown s(v)(u-v) - \frac{1}{2\alpha}(u-v)^T \bigtriangledown B(v)(u-v)(u-v)$$

$$\leq -(\lambda_{min} + \frac{\beta_{min}}{2\alpha})||d||_2^2. \quad \square$$

In practice, we may expect that the form (17) provides a steeper descent direction.

Proposition 3.7 *Assume that $\bigtriangledown s(v)$ is positive definite on Ω_v and that $\bigtriangledown_u \phi(u,v) = -\bigtriangledown_v \phi(u,v)$. Then there is only one stationary point of f which is the unique solution of VIP_v, i.e.,*

$$f'(v^*; v - v^*) \geq 0 \text{ for all } v \in \Omega_v$$

$$\Rightarrow \quad v^* \text{ is the unique solution of } VIP_v.$$

Proof. First we prove that any stationary point of f is a solution of VIP_v. Suppose $f'(v^*; v - v^*) \geq 0$ holds for all $v \in \Omega_v$. We have:

$$\sup_{u \in H(v^*)} (v-v^*)^T[s(v^*) - \bigtriangledown s(v^*)(u-v^*) - \frac{1}{\alpha} \bigtriangledown_v \phi(u, v^*)] \geq 0$$

Letting $v = u$ and rearranging, we obtain, from (19),

$$(u-v^*)^T \bigtriangledown s(v^*)(u-v^*) \leq [s(v^*) + \frac{1}{\alpha} \bigtriangledown_u \phi(u, v^*)]^T(u-v^*) \leq 0$$

which implies $u = v^*$ since $\bigtriangledown s(v)$ is positive definite on Ω_v. By Proposition 3.3, we conclude that v^* is a solution of VIP_v.

It is well known, however, that if $\bigtriangledown s(v)$ is positive definite, then VIP_v has a unique solution. Suppose that there are two stationary points. Then as in the first part, the two stationary points must be solutions of VIP_v, which is a contradiction. $\square$

Proposition 3.8 *Assume that $s(v)$ is monotone on Ω_v,*
$\bigtriangledown_u \phi(u,v) = -\bigtriangledown_v \phi(u,v)$ and $0 < \alpha < \infty$. If v^* is a solution of VIP_v, then v^* is a stationary point of $f(v)$ over Ω_v.

Proof. Since $0 < \alpha < \infty$ and (13) is satisfied, subproblem (18) is well defined and its solution is unique, i.e., $u = H(v^*) = v^*$. We have

$$f'(v^*; u - v^*) = (u - v^*)^T[s(v^*) - (\nabla s(v^*)(u - v^*) - \frac{1}{\alpha} \nabla_v \phi(u, v^*))]$$

$$= s(v^*)^T(u - v^*).$$

Since $s(v^*)^T(u - v^*) \geq 0$, we have $f'(v^*; u - v^*) \geq 0$. □

Proposition 3.9 *Assume that $s(v)$ is strongly monotone, $0 < \alpha < \infty$, the form (17) of $\phi(u, v)$ is used and $\nabla B(v)(H(v) - v)$ is positive semi-definite on Ω_v. Then v^* is a solution of VIP_v, if and only if v^* is a stationary point of $f(v)$ over Ω_v.*

Proof. In this case, subproblem (18) is well defined and its solution is unique. Suppose that v^* is a solution of VIP_v. We have $H(v^*) = v^*$ and

$$f'(v^*; v - v^*) = (v - v^*)^T[s(v^*) - (\nabla s(v^*) - \frac{B(v^*)}{\alpha})^T(H(v^*) - v^*)$$

$$-\frac{1}{\alpha}(H(v^*) - v^*)^T \nabla B(v^*) - v^*)]$$

$$= s(v^*)^T(v - v^*) \geq 0 \text{ for all } v \in \Omega_v.$$

Conversely, we have

$$f'(v^*; v - v^*) \geq 0 \text{ for all } v \in \Omega_v.$$

Letting $v = H(v^*)$ in the above expression and rearranging, we have

$$(H(v^*) - v^*)^T \nabla s(v^*)(H(v^*) - v^*)$$

$$+\frac{1}{\alpha}(H(v^*) - v^*)^T \nabla B(v^*)(H(v^*) - v^*)(H(v^*) - v^*)$$

$$\leq [s(v^*) + \frac{B(v^*)}{\alpha}(H(v^*) - v^*)]^T(H(v^*) - v^*)$$

$$\leq 0.$$

This implies that $H(v^*) = v^*$, since $\nabla s(v^*)$ is positive definite on Ω_v and $\nabla B(v^*)(H(v^*) - v^*)$ is positive-semi definite Ω_v. By Proposition 3.3, v^* is a solution of VIP_v. □

Following the analysis above, we are ready to introduce the following general descent framework for VIP_v.

Proposition 3.10 *Let $\{v^l\}$ be a sequence generated by the iteration*

$$v^{l+1} = v^l + t_l d^l, \quad l = 0, 1, 2, ... \tag{29}$$

where $d^l = u^l - v^l (u^l \in H(v^l))$ is a descent direction of f at v^l and $t_l \in [0, 1]$ is determined from

$$f(v^l + t_l d^l) = \min\{f(v^l + td^l) \quad | \quad 0 \leq t \leq 1\}. \tag{30}$$

Suppose that Ω_v is compact and convex, the mapping $s(v) : \Omega_v \subset R^m \to R^m$ is continuously differentiable and $\nabla s(v)$ is positive definite on Ω_v. Furthermore, assume that $\nabla_u \phi(u, v) = -\nabla_v \phi(u, v)$ or $\nabla B(v)(H(v) - v)$ is positive definite on Ω_v. Then, for any starting point $v^0 \in \Omega_v$, the generated sequence $\{v^l\}$ lies in Ω_v and converges to a unique solution of VIP_v.

Proof. Since the points v^l and $H(v^l) = v^l + d^l$ both belong to Ω_v and since $0 \leq t_l \leq 1$, it follows from the convexity of Ω_v that the sequence $\{v^l\}$ lies in Ω_v. By compactness of Ω_v, all limit points of the sequence $\{v^l\}$ stay in Ω_v.

To prove convergence of the algorithm, we use Zangwill's Convergence Theorem A (Zangwill, 1969). The assumption that Ω_v is compact ensures that Condition 1 of Convergence Theorem A holds. Condition 2 (the descent property) is ensured also by Propositions 3.5 and 3.6. Now consider Condition 3, i.e., upper semicontinity of the algorithmic mapping

$$A(v^l) = M(v^l)D(v^l)$$

where

$M(v^l)$ is the search map;
$D(v^l)$ is the direction map, i.e.,
$D(v^l) = (v^l, d^l) = (v^l, u^l - v^l)$ where $u^l \in H(v^l)$ and $d^l = u^l - v^l$.

We first study the direction map. Let $v^l \to v^\infty$ for a subsequence l and $d^l \to d^\infty$ for the subsequence l. Then

$$u^l \to u^\infty = d^\infty - v^\infty.$$

To prove that $D(v)$ is upper semicontinuous, it is sufficient to establish that $u^\infty \in H(v^\infty)$. Then, because $d^\infty = u^\infty - v^\infty$, we have

$$(v^\infty, d^\infty) \in D(v^\infty).$$

Since all u^l are feasible, u^∞ is also feasible. Moreover, by the definition of u^l, we have

$$[s(v^l) + \frac{1}{\alpha} \nabla_u \phi(u^l, v^l)]^T (u - u^l) \geq 0, \quad \text{for all} \quad u \in \Omega_v.$$

Since $\phi(u, v)$ is continuously differentiable by definition,

$$[s(v) + \frac{1}{\alpha} \nabla_u \phi(u, v)]$$

is upper semicontinuous.

Taking limits in the above inequality, we get

$$[s(v^\infty) + \frac{1}{\alpha} \nabla_u (u^\infty, v^\infty)]^T (u - u^\infty) \geq 0, \quad \text{for all} \quad u \in \Omega_v.$$

Then $u^\infty \in H(v^\infty)$ and $D(v)$ is upper semicontinuous.

It is well known that many search algorithms such as (30) induce the upper semicontinuity of $M(v)$. In this case, $A(v)$ is also upper semicontinuous.

We conclude that the algorithmic map is upper semicontinuous and consequently $f(v^l) \to 0$. This implies by Theorem 3.1 that $\{v^l\}$ is convergent to a solution of VIP_v which is unique since $s(v)$ is strongly monotone. All the other subsequences are also convergent to the solution. □

We note that the proposed algorithm is difficult to implement due to the fact that the evaluation of $f(v)$ involves an exact minimization, difficult to perform unless Ω_v has very special structure. Due to the fact, the exact line search is even more costly. An alternative approach for the line search would be to carry out an inexact line search, such as, Armijo-type step rule (Ortega and Rheinboldt, 1970). If $\phi(u, v)$ is of the form (16), then the method is PBOM (see Fukushima, 1989). If $\phi(u, v)$ takes the form (17), we shall refer to the resulting algorithm as the Linearized Jacobi based method.

When $\alpha = \infty$, the existence of the descent direction is ensured by the results of Proposition 3.5 and Proposition 3.6. The actual computation of the descent direction for this case is not trivial, as studied by Bertsekas and

Mitter (1973) and Marcotte (1985); it involves the solution of a suitable mathematical programming problem.

We close this section with the following equivalence results.

Proposition 3.11 *A solution v to VIP_v can be obtained by solving either one of the following equivalent problems:*

1) $s(v)^T(u-v) \geq 0$, for all $u \in \Omega_v$.

2) $\max\limits_{v\in\Omega_v} \min\limits_{u\in\Omega_v} s(v)^T(u-v) + \frac{1}{\alpha}\phi(u,v)$.

3) $\min -s(v)^T(u-v) - \frac{1}{\alpha}\phi(u,v)$

$$\text{s.t.} \quad v \in \Omega_v$$

$$u \in \arg\min s(v)^T(u-v) + \frac{1}{\alpha}\phi(u-v)$$

$$\text{s.t.} \quad u \in \Omega_v$$

4) $\max\limits_{v\in\Omega_v} \min\limits_{u\in\Omega_v} s(v)^T(u-v)$.

5) $\min -s(v)^T(u-v)$

$$\text{s.t.} \quad v \in \Omega_v$$

$$u \in \arg\min s(v)^T(u-v)$$

$$\text{s.t.} \quad u \in \Omega_v$$

We note that 3) and 5) are bilevel optimization problems.

Proof.

1) $\Leftrightarrow$ 3). This is assured by Theorem 3.1.

2) $\Leftrightarrow$ 3). 2) is a reformulation of 3).

1) $\Leftrightarrow$ 4). This is assured by Theorem 3.1 when $\alpha = \infty$.

5) $\Leftrightarrow$ 4). 5) is only a reformulation of 4). □

From the above Proposition, we note that the variational inequality problem is a special case of the max-min problem while the max-min problem is a special case of the bilevel problem.

4 Various realizations of the general algorithmic framework

In this section, we consider various realizations of the general descent framework suggested by Proposition 3.10. This analysis serves as a bridge connecting some algorithms of both the iterative methods and optimization methods.

Sufficient conditions for convergence of the following algorithms are not presented here. The interested reader is referred to the corresponding references.

Proposition 4.1 *If* $0 < \alpha < \infty$ *and* $t_l = 1$, *and* $\phi(u, v)$ *is given by*

$$\phi(u, v) = \frac{1}{2}(u - v)^T B(v)(u - v),$$

then the algorithm corresponds to a family of iterative methods already discussed. As particular instances, we obtain

1) the symmetrized Newton method where $B(v) = \frac{\nabla s(v) + \nabla s(v)^T}{2}$ *(Hammond, 1984);*

2) the Quasi-Newton methods where $B(v)$ *is a symmetric positive definite matrix of approximation of* $\nabla s(v)$ *(Josephy, 1979);*

3) the Linearized Jacobi method where $B(v)$=$Diag(\nabla s(v))$ *(Wu, Florian and Marcotte, 1990);*

4) the projection method where $B(v)$ *is a positive symmetric definite matrix* $\overline{B}$ *(Dafermos, 1980, Bertsekas, 1982).*

Note that these four methods are not descent algorithms in nature.

Proposition 4.2 *If* $\nabla s(v)$ *is symmetric, then* VIP_v *is equivalent to a convex minimization program. If the forms (15), (16) or (17) of* $\phi(u, v)$ *are used and* $0 < \alpha < \infty$, *then the algorithm reduces to Migdalas's regularized Frank-Wolfe method.*

Proof. See Migdalas (1990). □

Proposition 4.3 *Assume that* $\nabla s(v) = D$ *where* D *is symmetric, positive definite and constant. If* $\phi(u, v)$ *is defined as (16) where* B *is diagonal with positive entries and furthermore* Ω_v *consists of one constraint and bounded variables, then the algorithm is equivalent to Dussault-Ferland-Lemaire's method (DFL).*

Proof. See Dussault, Ferland and Lemaire (1986). In this case, VIP_v represents the first-order optimality condition of a quadratic convex program such as

$$(SS) \qquad \min \; g(v) = \frac{1}{2} v^T D v + c^T v, \quad \text{s.t.} \quad v \in \Omega_v$$

where D is a symmetric and positive definite matrix, c is a vector and $s(v) = v^T D + c^T$.

In DFL, given a feasible v, a separable subproblem of the form:

$$\min \frac{1}{2} u^T B u + C^T u$$

$$\text{s.t.} \; u \in \Omega_v$$

where $C = c + (D - B)v$ is solved. Thus, we need only to show that this separable subproblem is equivalent to the subproblem defined in (18) if the form (16) of $\phi(u, v)$ is taken and $\alpha = 1$. Consider the corresponding objective function in (18):

$$s(v)^T(u - v) + \frac{1}{2}(u - v)^T B(u - v)$$

$$= (D^T v + c^T)(u - v) + \frac{1}{2}(u^T B u + v^T B v - 2u^T B v)$$

$$= \frac{1}{2} u^T B u + [c + (D - B)v]^T u + [\frac{1}{2} v^T B v - s(v)^T v]$$

$$= \frac{1}{2} u^T B u + C^T u + \frac{1}{2} v^T B v - s(v)^T v$$

which is equivalent to the separable problem (SS). □

Proposition 4.4 *If $\alpha = 1$ and the form (16) of $\phi(u, v)$ is used, then the algorithm reduces to Fukushima's algorithm.*

Proof. Fukushima (1990) considers the form (16). □

5 An inexact method

As indicated in Proposition 3.10, a convex optimization subproblem must be solved at each iteration. In general, the problem can not be solved exactly. We will therefore, in this section, propose an inexact scheme that retains the global convergence property possessed by the previous algorithm. It is noted that the inexact scheme may offer a a good trade-off between the amount of work required per iteration and the total number of iterations carried out. In the following, we restrict our analysis to the case where $0 < \alpha < \infty$.

Define

$$\epsilon_l = \min\{\frac{1}{l}, \epsilon||d^l||_2^2\} \tag{31}$$

where ϵ_l is an error term at iteration l, d^l is a descent direction at iteration l and ϵ is a positive parameter.

We have the following in an obvious manner:

1) $\epsilon_{l+1} < \epsilon_l$

2) $\epsilon_l \leq \epsilon||d^l||_2^2$

3) $\epsilon_l \leq \frac{1}{l}$

4) $\lim_{l\to\infty} \epsilon_l = 0.$

Let $\hat{u}$ satisfy the inequality:

$$[s(v^l) + \frac{1}{\alpha} \nabla_u \phi(\hat{u}, v^l)]^T (v - \hat{u}) \geq -\epsilon_l, \quad \text{for all} \quad v \in \Omega_v, \tag{32}$$

i.e., $\hat{u}$ is an ϵ_l-optimal solution of the subproblem (18).

Proposition 5.1 *Let $u = H(v^l)$ and $d^l = \hat{u} - v^l$. If $\phi(u, v)$ is strongly convex in u, i.e., there exists a positive constant γ_{min} such that*

$$[\nabla_u \phi(u', v) - \nabla_u \phi(u'', v)]^T [u' - u''] \geq \gamma_{min}||u' - u''||_2^2, \tag{33}$$

then we have

$$||\hat{u} - u||_2 \leq \sqrt{\frac{\epsilon\alpha}{\gamma_{min}}}||d^l||_2. \tag{34}$$

Proof. By definition, u satisfies

$$[s(v^l) + \frac{1}{\alpha} \nabla_u \phi(u, v^l)]^T (v - u) \geq 0 \quad \text{for all} \quad v \in \Omega_v. \tag{35}$$

Letting $v = u$ in (32), $v = \hat{u}$ in (35), and adding (32) and (35), we obtain, after rearranging terms:

$$[\nabla_u \phi(\hat{u}, v^l) - \nabla_u \phi(u, v^l)]^T [u - \hat{u}] \geq -\epsilon_l \alpha$$

$$||\hat{u} - u||_2^2 \leq \frac{\epsilon_l \alpha}{\gamma_{min}} \qquad \text{using (33)}$$

$$\leq \frac{\epsilon \alpha}{\gamma_{min}} ||d^l||_2^2 \qquad \text{using} \qquad 2) \text{ above.}$$

Hence

$$||\hat{u} - u||_2 \leq \sqrt{\frac{\epsilon \alpha}{\gamma_{min}}} ||d^l||_2. \quad \square$$

Proposition 5.2 *Assume that*

1) $0 < \alpha < \infty$ (36)

2) $\nabla_u \phi(u, v) = -\nabla_v \phi(u, v)$ (37)

3) $\nabla_u \phi$ *is strongly monotone on* Ω_v (38)

4) $\nabla_u \phi$ *is Lipschitz continuous on* Ω_v, *with Lipschitz constant* L, *i.e.,*

$$||\nabla_u \phi(u', v) - \nabla_u \phi(u'', v)||_2 \leq L||u' - u''||_2 \tag{39}$$

5) $\epsilon + \lambda_{max} \sqrt{\frac{\epsilon \alpha}{\gamma_{min}}} + L \sqrt{\frac{\epsilon}{\gamma_{min} \alpha}} < \lambda_{min}$ (40)

where λ_{min} *and* λ_{max} *are the minimum and maximum eigenvalues of* $\nabla s(v)$ *respectively. If* $d^l = \hat{u} - v^l \neq 0$ *where* $\hat{u}$ *satisfies (32), then* d^l *is a descent direction of* $f(v)$ *at* v^l, *i.e.,* $f'(v^l; d^l) < 0$.

Proof. Let $u = H(v)$. We have

$$f'(v^l; d^l) = (\hat{u} - v^l)^T [s(v^l) - \nabla s(v^l)(u - v^l) - \frac{1}{\alpha} \nabla_v \phi(u, v^l)]$$

$$= [s(v^l) - \frac{1}{\alpha} \nabla_v \phi(u, v^l)]^T(\hat{u} - v^l) - (\hat{u} - v^l)^T \nabla s(v^l)(u - v^l)$$

$$= A1 + A2$$

where

$$A1 = [s(v^l) - \frac{1}{\alpha} \nabla_v \phi(u, v^l)]^T(\hat{u} - v^l)$$

$$A2 = -(\hat{u} - v^l)^T \nabla s(v^l)(u - v^l)$$

Consider $A1$:

$$A1 = [s(v^l) - \frac{1}{\alpha} \nabla_v \phi(u, v^l)]^T(\hat{u} - v^l)$$

$$= [s(v^l) + \frac{1}{\alpha} \nabla_u \phi(u, v^l)]^T(\hat{u} - v^l) \qquad \text{by (37)}$$

$$= -[s(v^l) + \frac{1}{\alpha} \nabla_u \phi(\hat{u}, v^l)]^T(v^l - \hat{u})$$

$$+\frac{1}{\alpha}[\nabla_u \phi(u, v^l) - \nabla_u \phi(\hat{u}, v^l)]^T(\hat{u} - v^l)$$

$$\leq \epsilon_l + \frac{1}{\alpha}[\nabla_u \phi(u, v^l) - \nabla_u \phi(\hat{u}, v^l)]^T(\hat{u} - v^l) \qquad \text{by (32)}$$

$$\leq \epsilon ||d^l||_2^2 + \frac{1}{\alpha} L ||u - \hat{u}||_2 \cdot ||d^l||_2 \qquad \text{by (39)}$$

$$\leq (\epsilon + L\sqrt{\frac{\epsilon}{\gamma_{min}}}\alpha)||d^l||_2^2 \qquad \text{by (34)}$$

Now consider $A2$:

$$A2 = -(\hat{u} - v^l)^T \nabla s(v^l)(u - v^l)$$

$$= -(\hat{u} - v^l)^T \nabla s(v^l)(\hat{u} - v^l) - (\hat{u} - v^l)^T \nabla s(v^l)(u - \hat{u})$$

$$\leq -\lambda_{min}||d^l||_2^2 - (\hat{u} - v^l)^T \nabla s(v^l)(u - \hat{u})$$

$$\leq -\lambda_{min}||d^l||_2^2 + ||d^l||_2 \cdot ||\nabla s(v^l)||_2 \cdot ||\hat{u} - u||_2$$

$$\leq -\lambda_{min}||d^l||_2^2 + \lambda_{max}||d^l||_2\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}}||d^l||_2 \qquad \text{by (34)}$$

$$= (-\lambda_{min} + \lambda_{max}\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}})||d^l||_2^2$$

Thus

$$f'(v^l;d^l) \leq (\epsilon + L\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}} - \lambda_{min} + \lambda_{max}\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}})||d^l||_2^2 < 0. \qquad \text{by (40).} \quad \square$$

Proposition 5.3 *Assume that*

1) $\phi(u,v) = \frac{1}{2}(u-v)^T B(v)(u-v)$;

2) $\nabla B(v)(u-v)$ *is positive definite with minimum eigenvalue* β_{min};

3) (33) is satisfied;

4) $B(v)$ *is a symmetric and positive definite matrix with maximum eigenvalue* θ_{max};

5) $0 < \alpha < \infty$;

6) $\epsilon + (\frac{\theta_{max}}{\alpha} - \lambda_{min} - \frac{\beta_{min}}{2\alpha})\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}} < \lambda_{min} + \frac{\beta_{min}}{2\alpha_{min}}$.

If $d^l = \hat{u} - v^l \neq 0$ *where* $\hat{u}$ *satisfies (32), then* d^l *is a descent direction for* $f(v)$ *at* v^l, *i.e.,*

$$f'(v^l;d^l) < 0.$$

Proof. Let $u = H(v^l)$. We have:

$$f'(v^l;d^l) = (\hat{u}-v^l)^T[s(v^l) - (\nabla s(v^l) - \frac{B(v^l)}{\alpha})(u-v^l)$$

$$-\frac{1}{2\alpha}(u-v^l)^T \nabla B(v^l)(u-v^l)]$$

$$= [s(v^l) - (\nabla s(v^l) - \frac{B(v^l)}{\alpha})(u-v^l)]^T(\hat{u}-v^l)$$

$$-\frac{1}{2\alpha}[(u-v^l)^T \nabla B(v^l)(u-v^l)]^T(\hat{u}-v^l)$$

$$= [s(v^l) + \frac{B(v^l)}{\alpha}(u-v^l)]^T(\hat{u}-v^l) - (u-v^l)^T \nabla s(v^l)(\hat{u}-v^l)$$

$$-\frac{1}{2\alpha}[(u-v^l)^T \nabla B(v^l)(u-v^l)]^T(\hat{u}-v^l)$$

$$= [s(v^l) + \frac{B(v^l)}{\alpha}(\hat{u} - v^l) + \frac{B(v^l)}{\alpha}(u - \hat{u})]^T(\hat{u} - v^l)$$

$$-(\hat{u} - v^l) \bigtriangledown s(v^l)(\hat{u} - v^l) - (u - \hat{u})^T \bigtriangledown s(v^l)(\hat{u} - v^l)$$

$$-\frac{1}{2\alpha}[(\hat{u} - v^l)^T \bigtriangledown B(v^l)(u - v^l)]^T(\hat{u} - v^l)$$

$$-\frac{1}{2\alpha}[(u - \hat{u})^T \bigtriangledown B(v^l)(u - v^l)]^T(\hat{u} - v^l)$$

$$\leq \epsilon_l - \lambda_{min}||d^l||_2^2 - \frac{1}{2\alpha}\beta_{min}||d^l||_2^2$$

$$+(u - \hat{u})^T\frac{B(v^l)}{\alpha}(\hat{u} - v^l) - (u - \hat{u})^T \bigtriangledown s(v^l)(\hat{u} - v^l)$$

$$-\frac{1}{2\alpha}(u - \hat{u})^T \bigtriangledown B(v^l)(u - v^l)(\hat{u} - v^l) \qquad \text{by (32) and 2)}$$

$$\leq (\epsilon - \lambda_{min} - \frac{1}{2\alpha}\beta_{min})||d^l||^2 + (\frac{\theta_{max}}{\alpha} - \lambda_{min} - \frac{\beta_{min}}{2\alpha})||\hat{u} - u||_2 \cdot ||d^l||_2$$

$$\leq [\epsilon - \lambda_{min} - \frac{1}{2\alpha}\beta_{min} + (\frac{\theta_{max}}{\alpha} - \lambda_{min} - \frac{\beta_{min}}{2\alpha})\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}}] \cdot ||d^l||_2^2$$

< 0 by 6) □

An important issue is how to find efficiently $\hat{u}$ satisfying (32).

As we already know, if $\epsilon = 0$, then condition (32) is the optimality condition of the subproblem (18), i.e.,

$$\min \psi(u, v^l) \tag{41}$$

$$\text{s.t. } u \in \Omega_v. \tag{42}$$

Let the optimal solution be u^*. By convexity and differentiability of ψ, $\hat{u}$ is an ϵ-optimal solution to (42), i.e., $\psi(u^*) + \epsilon \geq \psi(\hat{u})$.

Motivated by the network equilibrium problem where Ω_v is polyhedral and for which a linear program can easily be solved, we use the linear approximation method (Frank-Wolfe) to solve (41) and (42).

Now consider the linear approximation problem for a given $\hat{u}$

$$\Phi = \min \bigtriangledown_u \psi(\hat{u}, v^l)(u - \hat{u})$$

s.t. $u \in \Omega_v$, (43)

whose optimal solution is $\overline{u}$ and where $\nabla_v \psi(\hat{u}, v^l) = s(v^l) + \frac{1}{\alpha} \nabla_u \phi(\hat{u}, v^l)$. If $\Phi \geq -\epsilon_l$, then $\hat{H}(v^l) = \hat{u}$. Otherwise $d = \overline{u} - \hat{u}$ is a descent direction of $\psi(u, v^l)$ at $\hat{u}$. Then find a stepsize α^* such that

$$\alpha^* = \arg \min_{0 \leq \alpha \leq 1} \{\psi(\hat{u} + \alpha d, v^l)\} \tag{44}$$

Let $\hat{u} = \hat{u} + \alpha^* d$. Repeat the above procedure again. From the above discussion, we can see that subproblem (18) is solved by performing a sequence of linear programs. Now we are ready to present the inexact algorithm A for the VIP_v as follows.

Algorithm A

[step 1] [Initialization] Find $v^0 \in \Omega_v$ and set $l = 0$;

[step 2] [Subproblem]

[step 2.1] [Initialization] Let $u^0 = v^l$ and set $k = 0$;

[step 2.2] [Linear program] $\Phi = \min_{u \in \Omega_v} \nabla_u \psi(u^k, v^l)(u - u^k)$;
Let the optimal solution be $\overline{u}$.

[step 2.3] [Test] Let $\epsilon_l = \min\{\frac{1}{l}, \epsilon ||d^l||_2^2\}$.
If $\Phi \geq -\epsilon_l$,
then $\hat{u} = u^k$. Goto Step 3;
else $d = \overline{u} - u^k$;

[step 2.4] [Line search] $\alpha^* = \arg\min_{0 \leq \alpha \leq 1} \{\psi(u^k + \alpha d, v^l)\}$;

[step 2.5] [Update] $u^{k+1} = u^k + \alpha^* d$, $k = k + 1$.
Goto Step 2.2;

[step 3] [Descent direction] $d^l = \hat{u} - v^l$;

[step 4] [Stopping condition] If $||d^l||_2 \approx 0$, then stop. The inexact solution is $\hat{u}$.

[step 5] [Line search] $\alpha_l = \arg\min_{0 \leq \alpha \leq 1} f(v^l + \alpha d^l)$;

[step 6] [Update] $v^{l+1} = v^l + \alpha_l d^l$; $l = l + 1$. Goto Step 2.

Proposition 5.4 *Assume that Ω_v is compact and convex, the mapping $s(v)$: $\Omega_v \subset R^m \to R^m$ is continuously differentiable, $\nabla s(v)$ is positive definite on Ω_v and, either $\nabla_u \phi(u,v) = -\nabla_v \phi(u,v)$ or $\nabla B(v)(u-v)$ is positive definite on Ω_v. If conditions in Propositions 5.1, 5.2 and 5.3 are satisfied, then from an arbitrary point $v^0 \in \Omega_v$, the sequence $\{v^l\}$ generated by the Algorithm A lies in Ω_v and converges to the unique solution of VIP_v.*

Proof. The proof will be similar to that of Proposition 3.10, if we can show that [step 2] in Algorithm A is well defined and convergent. In fact, the method for solving the subproblem in [step 2] is the application of the Frank-Wolfe method. Since $\Phi(u,v)$ is continuously differentiable and convex on u and Ω_v is closed and bounded, then according to Martos (1975), the sequence $\{u^k\}$ generated in [Step 2] converges globally to an ϵ-optimal solution to (41) and (42). □

If $\nabla s(v)$ is symmetric for all $v \in \Omega_v$, then Algorithm A reduces to Migdalas's method (1990).

6 Conclusion

In this paper, we developed a general descent framework for the monotone variational inequality VIP_v and analysed its relationship with some existing algorithms. Based on the analysis, we provided an inexact algorithm and proved that it converges globally under certain conditions.

It should be noticed that a decomposition approach can be applied to Algorithm A. In the case of the network equilibrium problem, the linear program in Step 2.2 can be decomposed into a set of independent linear subprograms, one per origin-destination pair. This decomposability is well suited to parallel processing. An obvious way to implement a parallel version is to assign one of the linear subprograms to each processor. Notice that this feature is not shared by Marcotte and Dussault's algorithm (1989) which requires the solution of a sequence of linear programs in the primal-dual space of variables.

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